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Topological Solitons and Zero Modes in High-T_c Superconductivity

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The realization of topological solitons, zero modes, and supersymmetry in the CP^1 nonlinear sigma model, a theory of high-T_c superconductivity, is presented.

1. INTRODUCTION

Despite intense effort on the high-temperature superconductors, there is no real agreement about how these fascinating materials are to be explained. They show apparently contradictory aspects. Many of the superconducting properties appear surprisingly similar to those of normal BCS superconductors, such as the $2e$ flux quantization (Gough *et al.*, 1987), measurements of penetration depth (Hashman *et al.*, 1989), Andreev reflection (Van Bentum *et al.*, 1988), and the Knight shift (Alloul *et al.*, 1988). But other experiments are surprisingly different, for example, the high value of transition temperature (Müller and Olsen, 1988), the short coherence length (Yamagishi *et al.*, 1988), and the remarkably small isotope effect (Faltens *et al.*, 1987). It is natural to suppose that phonons now are not responsible for high-T_c superconductivity.

Most new superconductors are all layer oxides, where each unit cell contains one or several parallel CuO_2 conducting planes (Müller and Olsen, 1988). Observed anisotropy indicates the quasi-two-dimensional character of CuO_2 planes. The stoichiometric CuO_2 planes form a nonconducting antiferromagnetic ordered structure. It is confirmed by Raman scattering, neutron scattering, and infrared optical experiment that the doping does not destroy the local moments. Still, there are low-energy spin excitations in the doped samples which are superconducting and there is strong correlation between hole and spin excitations. This leads to the magnon pairing mechanism (Birgeneau) or to a disordered RVB state (Anderson, 1987).

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It is interesting that the spin antiferromagnetic system in the continuum field approximation corresponds to the nonlinear σ model (Dzyaloshinskii *et al.*, 1988) in 2+1 dimensions (Wilczek and Zee, 1983) or to CP^1 nonlinear σ (Rajaramam, 1982) coupled to $SO(3)$ gauge field theory with a Chern–Simons (CS) term (Jackiw, 1984). This opens a broad window for applying the topological gauge field theory (Polyachronakos, 1987), which in 2+1 dimensions has exciting properties. In the presence of the CS term it leads to fractional charge (Niemi and Semenoff, 1986) and boson–fermion transmutation (Wilczek and Zee, 1983). In this way an exotic anyon (Yi-Hong Chen *et al.*, 1989) model appears in the high-temperature superconductivity.

The aim of this paper is to present the topological solitons and associated zero modes appearing in two-dimensional CuO_2 planes.

2. THE TOPOLOGICAL SOLITONS IN THE CP^1 SIGMA MODEL

The antiferromagnetic two-dimensional CuO_2 plane in the field theory limit may be described by the $SO(3)$ nonlinear sigma model (Mańka and Molak, 1989) with the Lagrange function

$$\mathscr{L}=\tfrac{1}{2}\partial_\mu \mathbf{n}^a\,\partial^\mu \mathbf{n}^a+\lambda(\mathbf{n}^a\mathbf{n}^a-1) \tag{1}$$

with $\mathbf{n}$ as the unit vector [$\mathbf{n}=(\sin\vartheta\cos\vartheta, \sin\vartheta\sin\vartheta, \cos\vartheta)$]. This leads to the Euler–Lagrange equations

$$\Box\mathbf{n}^a=-\lambda\mathbf{n}^a \tag{2}$$

and

$$\mathbf{n}^a\mathbf{n}^a=1 \tag{3}$$

with $\Box=-\partial_\mu\,\partial^\mu$. Using the unit vector condition (3), we get

$$(\Box-\mathbf{n}^b\,\Box\mathbf{n}^b)\mathbf{n}^a=0 \tag{4}$$

This equation can be solved using the Bogomolny ansatz

$$\partial_\mu\mathbf{n}=\mp\varepsilon_{\mu\nu}(\mathbf{n}\times\partial^\nu\mathbf{n}) \tag{5}$$

This produces the topological soliton (instanton in $D=1+1$ space-time) with the $S^2\to S^2$ mapping. As

$$\lim \mathbf{n}^a(x)=\mathbf{n}_0\in S^2 \tag{6}$$

then the $S^2\to S^2$ map defines the $\pi_2(S^2)=Z$ homotopy group, which gives us the topological charge

$$Q=\frac{1}{8\pi}\int d^2x\,\varepsilon_{\mu\nu}(\partial^\mu\mathbf{n}\times\partial^\nu\mathbf{n})=n \tag{7}$$

This situation is very similar to the case of the winding number for the vortex in superconductors [in this case we have the $\pi_1(S^1)=Z$ homotopy group]. The topological charge is responsible for the soliton energy quantization

$$E=4\pi Q=4\pi n \tag{8}$$

The soliton solution for the $SO(3)$ nonlinear sigma model is well known (Rajaraman, 1982). It may be described as

$$\omega(z)=\left[\frac{1}{\lambda}(z-z_0)\right]^n \tag{9}$$

where

$$z=x^1+x^2 \tag{10}$$

and

$$\omega=\omega^1+i\omega^2 \tag{11}$$

with $\lambda=a/2$. So, for example, for $n=1$ in the cylindrical coordinate system we have

$$n_r=\frac{2ar}{r^2+a^2} \tag{12}$$

$$n_3=\frac{r^2-a^2}{r^2+a^2} \tag{13}$$

$$n_\varphi=0 \tag{14}$$

where $r^2=x_1^2+x_2^2$ (Figure 1) (Mańka, 1989). Instead of the unit real vector $\mathbf{n}^a$, we may introduce the two complex scalar fields

$$\mathscr{Z}=\begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \tag{15}$$

via the identification

$$\mathbf{n}^i=-\mathscr{Z}^+\sigma^i\mathscr{Z} \tag{16}$$

with the restriction

$$\mathscr{Z}^+\mathscr{Z}=1 \tag{17}$$

σ_i are certainly the Pauli matrices. The Lagrangian must also include an extra term of the form $\lambda(\mathscr{Z}^+\mathscr{Z}-1)$ to account for the constraints. The $\mathbf{n}^i$ unit field now may be expressed by the field $\mathscr{Z}$ as

$$\mathbf{n}^1=-(z_1^*z_2+z_2^*z_1) \tag{18}$$

$$\mathbf{n}^2=-i(-z_1^*z_2+z_2^*z_1) \tag{19}$$

$$\mathbf{n}^3=-(z_1^*z_1-z_2^*z_2) \tag{20}$$

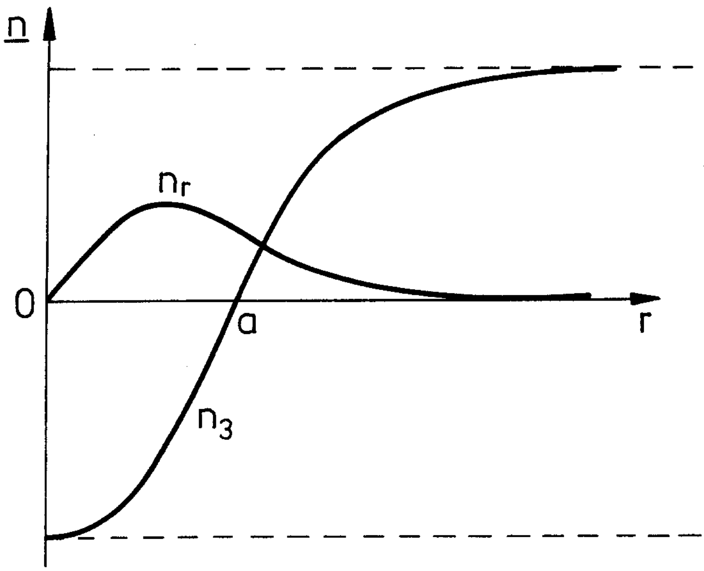

Fig. 1. The topological soliton profile.

The complex vector $\mathscr{Z}$ may be obtained from the $\mathscr{Z}_0 = \binom{0}{1}$ vector by the $SU(2)$ unitary transformation

$$\mathscr{Z} = U\mathscr{Z}_0 \tag{21}$$

where

$$U = e^{i\sigma^i \omega^i \vartheta/2} \in SU(2) \tag{22}$$

with

$$\omega^i = (-\sin\varphi, \cos\varphi, 0) \tag{23}$$

In the explicit form the vector is as follows:

$$\mathscr{Z} = \begin{Bmatrix} e^{-i\varphi} \sin\frac{1}{2}\vartheta \\ \cos\frac{1}{2}\vartheta \end{Bmatrix} \tag{24}$$

Let us notice that the transformation

$$\mathscr{Z} \to \mathscr{Z}' = e^{i\alpha}\mathscr{Z} \tag{25}$$

does not change the $\mathbf{n}^i$ vector. This means that we have identified the (z_1, z_2) and $(e^{i\alpha}z_1, e^{i\alpha}z_2)$ points. But this is just the CP^1 space definition. Now $\mathscr{L}_0$ given by (1) may be expressed by the complex $\mathscr{Z}$ field. We get

$$\mathscr{L}_0 = 2(\partial_\mu \mathscr{Z})^* \partial^\mu \mathscr{Z} + 2(\mathscr{Z}^* \partial_\mu \mathscr{Z})(\mathscr{Z}^* \partial^\mu \mathscr{Z}) + \lambda(\mathscr{Z}^* \mathscr{Z} - 1) \tag{26}$$

or equivalently

$$\mathscr{L}_0 = 2D_\mu \mathscr{Z}^* D^\mu \mathscr{Z} + \lambda(\mathscr{Z}^*\mathscr{Z} - 1) \tag{27}$$

with the covariant derivative as

$$D_\mu = \partial_\mu + iA_\mu \tag{28}$$

and the gauge field

$$A_\mu = i\mathscr{Z}^* \partial_\mu \mathscr{Z} \tag{29}$$

In this way, we can interpret the CP^1 sigma model as the $U(1)$ gauge field theory in the $e \to \infty$ limit. This means that

$$\mathscr{L}_0 = \lim_{e\to\infty} \mathscr{L}$$

where

$$\mathscr{L} = -\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + 2(D_\mu \mathscr{Z})^* D^\mu \mathscr{Z} + \lambda(\mathscr{Z}^*\mathscr{Z} - 1) \tag{30}$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu \tag{31}$$

As the kinetic Yang-Mills term is absent ($e \to \infty$), the only term which can appear is the pure topological Hopf (CS) term. The origin of this term has only topological nature. The reason lies in the fact that we have the vacuum degeneracy $|n\rangle$ numbered by the $\pi_2(S^2) = Z$ homotopy group. However, this degeneracy is destroyed by the quantum processes. The quantum tunneling processes produce the new vacuum

$$|\vartheta\rangle = \sum_n e^{-i\vartheta n}|n\rangle \tag{32}$$

Exactly in the same way, the quasimomentum state $|k\rangle$ is formed from the Wannier states $|n\rangle$. At the effective quasiclassical level the $|\vartheta\rangle$ state produces a new pure topological CS term,

$$\mathscr{L}_{\text{ef}} = \mathscr{L} + \mathscr{L}_{\text{CS}} \tag{33}$$

with

$$\mathscr{L}_{\text{CS}} = -\frac{1}{4\pi^2} \vartheta \varepsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} \tag{34}$$

The nonlinear sigma model with $\vartheta \neq 0$ represents a very interesting realization of the pure topological gauge field theory. Such theories have fascinating properties. They have no propagating degrees of freedom

(Witten, 1988*a*,*b*) and their Hilbert space is finite, possessing only a degenerate number of vacuum states with fractional charges (Polyachronakos, 1989). The $D=2+1$ topological gravity, which seems to be an interesting example, appears to be fully renormalizable (Witten, 1988*a*,*b*). The ϑ parameter may be regarded as a new parameter of the theory which should be established by experiment. If $\vartheta=0$, then the system is called the quantum paramagnetic state (QP), and if $\vartheta\neq 0$, the quantum spin liquid (QSL). If $\vartheta\neq 0$, then equations (33), (34) give us the Euler–Lagrange equations as follows:

$$-\partial_\mu F^{\mu\nu}=\tfrac{1}{2}m\,{}^*F^\nu \tag{35}$$

with

$${}^*F^\mu=\tfrac{1}{2}\varepsilon^{\mu\nu\rho}F_{\nu\rho} \tag{36}$$

and

$$m=\frac{e^2\vartheta}{\pi^2} \tag{37}$$

It is easy to show that the parameter m may be interpreted as the mass of the gauge field. For that purpose let us multiply the equation of motion (35) by $(mg_{\mu\nu}+\varepsilon_{\mu\rho\nu}\partial^\rho)$, using the fact that, according to equation (35), $\partial_\mu({}^*F^\mu)=0$, we get the Klein–Gordon equation

$$(\Box-m^2)({}^*F_\mu)=0 \tag{38}$$

This is the topological origin of the mass. We see that the field strength tensor propagates as a free field of mass m. If $e\to\infty$, then $n\to\infty$ and the gauge particles happen to be extremely heavy. In this way the propagating excitations disappear from the theory. However, the CS term has less symmetry than the kinetic Yang–Mills term. Due to the presence of $\varepsilon^{\mu\nu\rho}$, it violates discrete symmetries (T and P inverse symmetry). It is also not invariant with respect to the non-Abelian gauge group. The CS action

$$\mathcal{S}=\int d^3x\,\mathcal{L}_{\text{CS}} \tag{39}$$

transforms with respect to the gauge symmetry as follows:

$$\mathcal{S}_{\text{CS}}\to\mathcal{S}'_{\text{CS}}=\mathcal{S}_{\text{CS}}+2\vartheta Q_t \tag{40}$$

where $Q_t=n\in Z=\pi_2(S^2)$ is the topological charge (7). It is only invariant for trivial ($n=0$) gauge transformation. However, this is not so dangerous. In the Feynman path integration method the action appears as $\exp(i\mathcal{S})$. This term will be invariant if the appropriate quantization condition is imposed on ϑ. In this way the ϑ quantization appears and

$$\vartheta=0,\ \pi,\ \ldots \tag{41}$$

If $\vartheta \neq 0$, then the topological soliton can change its properties drastically. It gains the spin (Dunne *et al.*, 1989)

$$S_{\mathrm{sol}} = \frac{\vartheta}{2\pi} \tag{42}$$

In a similar way the $\mathscr{Z}$ quanta gain the spin

$$S_Z = \frac{\pi}{2\vartheta} \tag{43}$$

It is easy to notice that for $\vartheta = \pi$ we get $S_{\mathrm{sol}} = S_Z = 1/2$. This is the boson-fermion transmutation (Wilczek and Zee, 1983). In this case and for $n = 1$ the CS term can be fermionized at low momenta,

$$\exp\left[i \int d^3x \frac{1}{4\pi} \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho\right] = \int |\mathscr{D}\bar{\psi}\, \mathscr{D}\psi \exp\left[i \int d^3x\, \bar{\psi}\{i\gamma^\mu(\partial_\mu + iA_\mu) \pm m\}\psi\right] \tag{44}$$

In this way the neutral fermion (holino) appears in the theory. The effective Lagrange function now may be described as

$$\mathscr{L}_{\mathrm{ef}} = \mathscr{L}_0 + i\bar{\psi}\gamma^\mu(\partial_\mu + iA_\mu) \pm m\bar{\psi}\psi \tag{45}$$

Of course, it gives us the Dirac equation for the holino in the external gauge field A_μ,

$$i\gamma^\mu[(\partial_\mu + iA_\mu) \pm m]\psi = 0 \tag{46}$$

with $\gamma^0 = \sigma^3$, $\gamma^1 = i\sigma^1$, $\gamma^2 = i\sigma^2$.

3. ZERO MODES

The first high-T_c superconductors have very distinct properties. In the insulating phase, the strongly correlated Cu spins are assumed to be arranged antiferromagnetically (the homogeneous spin phase $n_3 = \mathrm{const}$). Although band calculations indicate a partially filled valence band of Cu($3d$)-O($2p$) hybridized orbitals, the planes are insulating due to the opening of a gap at the Fermi level accompanying the formation of the static spin-density wave (SDW) (spin Peierls transition; Figure 2). Holes in CuO_2 planes near the Fermi level may be described by the Dirac equation (with c as a material constant). The action is

$$\mathscr{L}_0 = i\psi\gamma^\mu\partial_\mu\psi \tag{47}$$

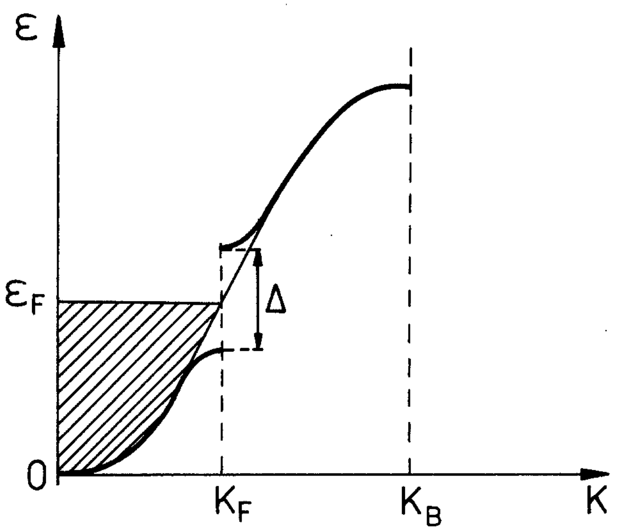

Fig. 2. The spin Peierls transition.

This means that near the Fermi level the dispersion relation is linear. Quasi-two-dimensionality means that CuO_2-plane fermions have only S_3 spin component,

$$S_3 = \tfrac{1}{2}\psi^+\sigma_3\psi = \tfrac{1}{2}\bar{\psi}\psi \tag{48}$$

The strong ($J_{\text{Cu-O}}$ 330 K ferromagnetic for $Y_1Ba_2Cu_3O_7$) interaction between hole spins and neighboring Cu orbitals may be described as

$$\mathscr{L}_I = \tfrac{1}{2}J\psi^+\sigma_3\psi(S\mathbf{n}_3) = \tfrac{1}{2}JS\mathbf{n}_3(x)\bar{\psi}\psi \tag{49}$$

In the field theory interpretation this is a generalization of the mass term in the Dirac equation. The total Lagrangian

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_I \tag{50}$$

leads to the antiferromagnetic Peierls transition. Indeed, in the case of a homogeneous phase ($\mathbf{n}_3 = 1$), equation (49) gives us the fermion mass

$$m = JS \tag{51}$$

which generates the gap Δ, as $E_k = (k^2 + m^2)^{1/2} = (k^2 + \Delta^2)^{1/2}$. In this case the metal state disappears. In the nonhomogeneous case in the presence of the topological solitons we have a more complicated situation. Now the solitons work rather as some potential in the Dirac equation

$$i\gamma^\mu\partial_\mu\psi + JS\mathbf{n}_3(x)\sigma_3\psi = 0 \tag{52}$$

Only at infinity when $\mathbf{n}_3(x) \to 1$ do we have the mass (51) interpretation. The Dirac equation (52) generates the Hamiltonian

$$H = \begin{Bmatrix} -m\mathbf{n}_3(r), & e^{-i\varphi}\partial_r \\ -e^{i\varphi}\partial_r, & m\mathbf{n}_3(r) \end{Bmatrix} \tag{53}$$

It is convenient to write the spinor ψ as

$$\psi = \begin{Bmatrix} e^{-i\varphi/2}u \\ e^{+i\varphi/2}v \end{Bmatrix} \tag{54}$$

This transforms the eigenvalue equation $H\psi = E\psi$ into $H'\psi' = E\psi'$, with

$$\psi' = \begin{Bmatrix} u \\ v \end{Bmatrix} \tag{55}$$

and

$$H' = \begin{Bmatrix} -m\mathbf{n}_3(r), & \partial_r \\ -\partial_r, & m\mathbf{n}_3(r) \end{Bmatrix} = i\sigma_2\partial_r - m\mathbf{n}_3(r)\sigma_3 \tag{56}$$

The unitary transformation $\mathcal{U} = \exp(-\frac{1}{2}\sigma_2\pi)$ transforms $H' \to H'' = \mathcal{U}^+ H' \mathcal{U}$ equal to

$$H'' = \begin{Bmatrix} 0, & \partial_r + W(r) \\ -\partial_r + W(r), & 0 \end{Bmatrix} = \sqrt{2}\, af^+ + \sqrt{2}\, a^+ f \tag{57}$$

or

$$H'' = \partial + \bar{\partial} \tag{58}$$

with $W(r) = m\mathbf{n}_3(r)$ as the superpotential. The operators ∂ and $\bar{\partial}$ realize the supersymmetric quantum mechanics. They are built from bosonic operators

$$a = \frac{1}{\sqrt{2}}[\partial_r + W(r)] \tag{59}$$

$$a^+ = \frac{1}{\sqrt{2}}[-\partial_r + W(r)] \tag{60}$$

obeying

$$[a, a^+] = W'(r) \tag{61}$$

and from fermionic operators

$$f = \begin{Bmatrix} 0, & 0 \\ 1, & 0 \end{Bmatrix} \tag{62}$$

$$f^+ = \begin{Bmatrix} 0, & 1 \\ 0, & 0 \end{Bmatrix} \tag{63}$$

obeying

$$\{f, f^{+}\} = 1 \tag{64}$$

$$f^2 = f^{+2} = 0 \tag{65}$$

In the result

$$\bar{\partial}^2 = \partial^2 = 0$$

The Laplace operator

$$\Delta = (H'')^2 = (\bar{\partial} + \partial)^2 = \bar{\partial}\partial + \partial\bar{\partial} \tag{66}$$

may be rewritten as

$$\Delta = 2\mathcal{H} \tag{67}$$

with

$$\mathcal{H} = \frac{1}{2}\left[\frac{d^2}{dr^2} + W^2(r)\right] + \frac{1}{2} W'(r)\sigma_3 \tag{68}$$

$\mathcal{H}$ is the Hamiltonian of the Witten supersymmetric quantum mechanics. $\{\mathcal{H}, \partial, \bar{\partial}\}$ are generators of supersymmetry. For example, taking $W(x) = x$, $\mathcal{H}$ describes the supersymmetric harmonic oscillator. It is physically realized in the two-dimensional case for electrons in an external magnetic field. The conditions

$$\partial\psi_0 = \partial\bar{\psi}_0 = 0 \tag{69}$$

define zero modes. Indeed, from (69) we have $\mathcal{H}\psi_0 = 0$ and $H\psi_0 = 0$. It is easy to notice that

$$\psi_0 \rightarrow \psi_0' = \psi_0 + \partial\alpha \tag{70}$$

where $\partial\bar{\partial}\alpha = 0$, defines equivalence classes. This is just the cohomology classes definition. The number of zero modes according to the Atijah–Hirzerbruch theorem has a topological origin. Equations (69) for zero modes have solution

$$\psi_0 = \begin{Bmatrix} 0 \\ v_0 \end{Bmatrix} \exp\left[-\int W(r)\, dr\right] \tag{71}$$

The zero mode chooses only one spin component. This is reminiscent of the chirality violation in higher-dimensional cases. For the supersymmetric harmonic oscillator, $\psi_0 = A \exp(\frac{1}{2}x^2)$ means simply the ground state now with energy $E = 0$.

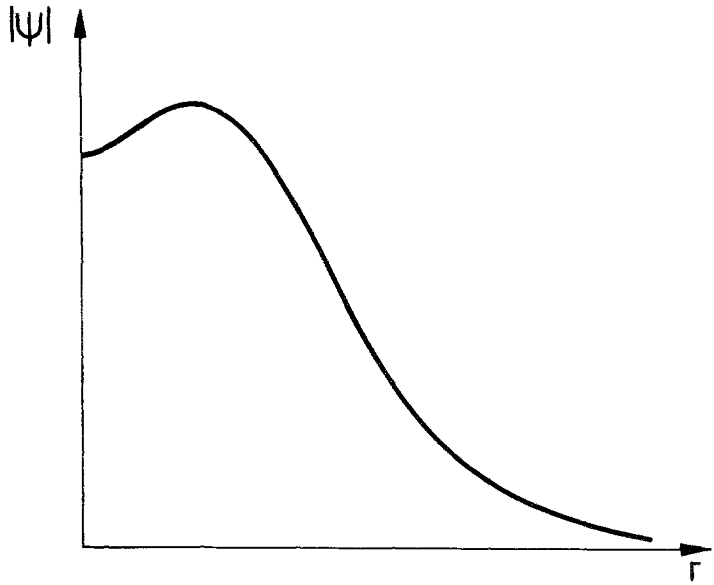

Fig. 3. The zero-mode profile.

The CP^1 topological soliton [Equations (21)-(24)] produces a zero mode which can be interpreted as a hole bound state captured on the soliton with energy $E = 0$ exactly (on the Fermi level). The soliton profile (13) gives

$$\psi_0 = \begin{Bmatrix} 0 \\ v_0 \end{Bmatrix} \exp\{m[r - 2a \operatorname{arctg}(x/a)]\} \tag{72}$$

This zero mode (Figure 3) will have an appropriate probability interpretation only if the hole spin is equal to $-1/2$. Spin $+1/2$ could not be captured inside the soliton. The s-pairing is possible only between a zero mode and the spin-1/2 hole. The bisoliton gives only the d-pairing. This is, however, rejected by the experiment (Alloul *et al.*, 1988).

4. CONCLUSION

In this paper a field-theoretical approach to high-T_c superconductivity was presented. The antiferromagnetic CuO_2 planes were represented by the CP^1 field theory. Topological solitons exist in this theory and allow one to realize topological field theory. Interactions between topological solitons and CuO_2-plane fermions leads to the soliton–fermion bound state (zero mode). A similar case occurs in one-dimensional polymers. Its charged solitons are responsible for the conductivity. In the result of the chiral anomaly they have a fractional charge. A similar phenomenon also should occur in the CP^1 case. It is interesting that the zero mode existence is a result of some supersymmetry. If the high-T_c superconductivity really is described by such a theory, then it seems to be quite interesting.

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